

STABILISATION OF THE LHS SPECTRAL SEQUENCE FOR ALGEBRAIC GROUPS

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ABSTRACT. In this note, we consider the Lyndon–Hochschild–Serre spectral sequence corresponding to the first Frobenius kernel of an algebraic group G and computing the extensions between simple G -modules. We state and discuss a conjecture that $E_2 = E_\infty$ and provide general conditions for low-dimensional terms on the E_2 -page to be the same as the corresponding terms on the E_∞ -page, i.e. its abutment.

1. INTRODUCTION

Let G be a reductive algebraic group over an algebraically closed field k of characteristic $p > 0$. Let λ and μ be two dominant weights for G . This paper concerns the representation theory of G and its first Frobenius kernel G_1 ; we refer to [Jan03] for notation. It is the purpose of this short note to state and provide some evidence towards the following conjecture.

Conjecture. *Suppose all G_1 -injective hulls have the structure of G -modules, for instance if $p \geq 2h - 2$. Then the Lyndon–Hochschild–Serre spectral sequence*

$$(*) \quad E_2^{ij} = \text{Ext}_{G/G_1}^i(k, \text{Ext}_{G_1}^j(L(\lambda), L(\mu))) \Rightarrow \text{Ext}_G^{i+j}(L(\lambda), L(\mu))$$

stabilises (i.e. reaches its abutment) at the E_2 -page. That is, $E_2^{ij} \cong E_\infty^{ij}$ for all i, j .

Hence

$$\text{Ext}_G^n(L(\lambda), L(\mu)) \cong \bigoplus_{i+j=n} \text{Ext}_{G/G_1}^i(k, \text{Ext}_{G_1}^j(L(\lambda), L(\mu))).$$

Note that it is an open conjecture of Humphreys and Verma that all G_1 -injective hulls do indeed have the structure of G -modules, possibly making the first hypothesis trivially satisfied.

Let us underline the fact that we are unaware of any occasion where any differential in the spectral sequence $(*)$ is known to be non-zero—even after replacing G with an arbitrary connected algebraic group and replacing $L(\lambda)$ and $L(\mu)$ by arbitrary G -modules. Showing that certain differentials in the spectral sequence are zero has some history; we pick out a few cases. For a large class of naturally occurring modules V and W , it was shown in [Par07] that when $G = \text{SL}_2$ the spectral sequence does stabilise at the E_2 -page. In particular the conjecture is confirmed for the case $G = \text{SL}_2$, with no condition on p . It was shown by Donkin in [Don82] that the differentials $d_{m,1} : E_2^{m,1} \rightarrow E_2^{m+2,0}$ are zero, also with no condition on p . Some other special cases involving maps needed to compute second cohomology were considered in work of McNinch [McN02], the second author [Ste10, Ste12], and Ibraev [Ibr11, Ibr12].

Another case in which the conjecture is true is if λ and μ are p -regular restricted weights, $p \geq 2h - 2$ and p is large enough that the Lusztig Character Formula holds. Then [PS13, Theorem 5.3] shows that the G -module $\text{Ext}_{G_1}^n(L(\lambda), L(\mu))^{[-1]}$ has a good filtration for each n . Under these

circumstances the spectral sequence moreover degenerates to a line; in particular the conjecture is true.

Note that the conjecture is not true if G is replaced by an arbitrary group. See [BF94, §6], [Lea93] and [Sie00] for examples of non-zero differentials.

The main theorem of this paper is a confirmation of the conjecture in a generic sense. Here, the vanishing of differentials of degree much lower than p is guaranteed.

Theorem. *Suppose $p \geq (r+1)(h-1)$. Then the differentials d_n^{ij} in the spectral sequence (*) satisfying $i \leq r-1$ and $n \geq 2$ or $j = 0$ and $n \geq 2$ or $j = 1$ and $n \geq 2$ are all zero.*

In particular,

$$\mathrm{Ext}_G^i(L(\mu), L(\lambda)) \cong \bigoplus_{j=0}^i E_2^{i-j,j}$$

for $i \leq r+1$.

We prove the above theorem by applying techniques from [Par07]. First, we show, in a proposition, that part of a minimal G_1 -injective resolution has a compatible G -structure. We then reconstruct the spectral sequence (*) in such a way that the bottom-most complex in the double complex giving the E_0 -page contains this part of a minimal G_1 -injective resolution. It follows that many maps in the E_0 -page are zero. Then some derived couple arguments prove the theorem.

2. PROPOSITION AND PROOF OF THE THEOREM

In the proposition below, note that the case $r = 0$ would be a special case of the Humphreys–Verma conjecture. (It is not known if the bound $p \geq 2h-2$ could be reduced to $p \geq h-1$ for G_1 -injective hulls to lift to G -modules.)

Proposition. *Let $r \geq 1$ and let $\mu \in X_1$. Provided $p \geq (r+1)(h-1)$, there is a minimal G_1 -resolution*

$$0 \rightarrow L(\mu) \rightarrow I_0 \rightarrow I_1 \rightarrow \cdots \rightarrow I_r \rightarrow \cdots$$

such that the sequence up to term I_r has a G -structure.

Proof. We prove *a fortiori* that there is such a sequence of G -modules with I_r having weights $\lambda = \lambda_0 + p\lambda_1$ with $\lambda_0 \in X_1$, which satisfy $(\lambda_1, \alpha_0^\vee) \leq (r+1)(h-1)$.

First, let us treat the case $r = 1$. Set $I_0 = Q_1(\mu)$. The hypotheses imply that $p \geq 2h-2$; thus we know that $Q_1(\mu)$ has the structure of a G -module. The injection $L(\mu) \rightarrow Q_1(\mu)$ is then a map of G -modules.

Let $M := Q_1(\mu)/L(\mu)$. We may write $\mathrm{Soc}_{G_1} M = \bigoplus_\nu L(\nu_0) \otimes M_\nu^F$ where $\nu_0 \in X_1$ and M_ν is some G -module. Set $I_1 = \bigoplus_\nu Q_1(\nu_0) \otimes M_\nu^F$. So $\mathrm{Soc}_{G_1} I_1 = \mathrm{Soc}_{G_1} Q_1(\mu)/L(\mu)$. (It is worth noting that the condition on the weights here is enough to ensure that $\mathrm{Soc}_{G_1} M = \mathrm{Soc}_G M$ but we do not need this fact explicitly.) Thus I_1 is the G_1 -injective hull of M , hence if there is a G -map $I_0 \rightarrow I_1$, this will be part of a minimal resolution. It remains to show that there is indeed a map $I_0 \rightarrow I_1$ of G -modules whose kernel is $L(\mu)$, i.e. a map $I_0/L(\mu) \rightarrow I_1$. Note that we do have a map $\mathrm{Soc}_{G_1} M \rightarrow I_1$ by construction, so consider the exact sequence

$$(*) \quad \mathrm{Hom}_G(M, I_1) \rightarrow \mathrm{Hom}_G(\mathrm{Soc}_{G_1} M, I_1) \rightarrow \mathrm{Ext}_G^1(M/\mathrm{Soc}_{G_1} M, I_1).$$

If we could show that the third term in this sequence is zero then we would have that the first map were surjective, hence the G -map $\text{Soc}_{G_1} M \rightarrow I_1$ would lift to a map $M = I_0/L(\mu) \rightarrow I_1$ and we would be done.

Now $\text{Ext}_G^1(M/\text{Soc}_{G_1} M, I_1)$ has a filtration by spaces $E = \text{Ext}_G^1(M/\text{Soc}_{G_1} M, Q_1(\nu_0) \otimes L(\nu_1)^F)$ over certain weights $\nu = \nu_0 + p\nu_1$. And E can be computed via the 5-term exact sequence of the LHS spectral sequence, of which part is

$$\begin{aligned} \text{Ext}_{G/G_1}^1(k, \text{Hom}_{G_1}(M/\text{Soc}_{G_1} M, Q_1(\nu_0) \otimes L(\nu_1)^F)) &\rightarrow E \\ &\rightarrow \text{Hom}_{G/G_1}(k, \text{Ext}_{G_1}^1(M/\text{Soc}_{G_1} M, Q_1(\nu_0) \otimes L(\nu_1)^F)). \end{aligned}$$

Now the third term here is zero, as $Q_1(\nu_0)$ is injective for G_1 , hence, to show $E = 0$, it suffices to show that the first term is zero.

Now $I_0 = Q_1(\mu)$, being a direct G_1 -summand of $St_1 \otimes L((p-1)\rho - \mu)$, has weights satisfying $\xi = \xi_0 + p\xi_1$ with $(\xi_1, \alpha_0^\vee) \leq h-1$. Thus the composition factors of I_0 (hence of M) are of the form $L(\xi_0) \otimes L(\xi_1)^F$. In particular, we have that the weights ν_1 satisfy $(\nu_1, \alpha_0^\vee) \leq h-1$. Thus $(\xi_1 + \rho, \alpha_0^\vee), (\nu_1 + \rho, \alpha_0^\vee) \leq 2h-2$ and our condition on p implies that they are both in the closure of the lowest alcove, $\bar{C}_\mathbb{Z}$. So let $L(\xi_0) \otimes L(\xi_1)^F$ be a composition factor of $M/\text{Soc}_{G_1} M$. We compute:

$$\begin{aligned} \text{Ext}_{G/G_1}^1(k, \text{Hom}_{G_1}(L(\xi_0) \otimes L(\xi_1)^F, Q_1(\nu_0) \otimes L(\nu_1)^F)) \\ \cong \text{Ext}_G^1(L(\xi_1), \text{Hom}_{G_1}(L(\xi_0), Q_1(\nu_0))^{[-1]} \otimes L(\nu_1)) \end{aligned}$$

Now $\text{Hom}_{G_1}(L(\xi_0), Q_1(\nu_0))$ is non-zero, thence equal to k , if and only $\xi_0 = \nu_0$; in that case, the term on the right becomes $\text{Ext}_G^1(L(\xi_1), L(\nu_1))$, and since $\xi_1, \nu_1 \in \bar{C}_\mathbb{Z}$, this vanishes by the linkage principle. This concludes the proof in case $r = 1$.

Now by induction we may assume that we have a sequence of G -modules

$$0 \rightarrow I_0 \rightarrow \cdots \rightarrow I_{r-2} \xrightarrow{\pi} I_{r-1},$$

which is minimal as an injective G_1 -resolution, such that the composition factors of I_{r-1} have high weights λ satisfying $\lambda = \lambda_0 + p\lambda_1$ with $\lambda_0 \in X_1$ and $(\lambda_1, \alpha_0^\vee) \leq r(h-1)$. We construct I_r in a similar way to before: set $I_r = \bigoplus_\nu Q_1(\nu_0) \otimes M_\nu^F$, where the sum is over the G -composition factors of $\text{Soc}_{G_1} I_{r-1}/\pi I_{r-2}$, where $\nu_0 \in X_1$ and a weight ν_1 of M_ν satisfies $(\nu_1, \alpha_0^\vee) \leq r(h-1)$. Thus a weight ξ of I_r , say $\xi_0 + p\xi_1$ with $\xi_0 \in X_1$ satisfies $(\xi_1, \alpha_0^\vee) \leq (\nu_1, \alpha_0^\vee) + (\rho, \alpha_0^\vee) = (r+1)(h-1)$ as required. Note that I_r is again a G_1 -injective hull of $I_{r-1}/\text{im } \pi$ so if we can show there is a G -module map $I_{r-1} \rightarrow I_r$ with kernel $\text{im } \pi$, we will be done.

Of course, it is equivalent to produce a map from $M := I_{r-1}/\text{im } \pi$ to I_r . By construction we do have a map from $\text{Soc}_{G_1} M \rightarrow I_r$. Now the same argument as before shows that the third term in the sequence (*) (with I_r replacing I_1) is zero. This completes the proof. \square

Proof of the theorem. We write $L(\mu) = L(\mu_0) \otimes L(\mu_1)^F$ using Steinberg's tensor product theorem where $\mu_0 \in X_1$ and $\mu_1 \in X^+$. Using the proposition we have a G -resolution which is also a G_1 -injective resolution:

$$0 \rightarrow L(\mu_0) \rightarrow I_0 \rightarrow I_1 \rightarrow \cdots \rightarrow I_r \rightarrow \cdots,$$

where, up to I_r , the resolution is minimal for G_1 .

We denote the differentials by $\delta_i : I_i \rightarrow I_{i+1}$ and the kernels by $K_i := \ker \delta_i$. Dimension shifting gives us $\text{Ext}_{G_1}^i(L(\lambda_0), L(\mu_0)) \cong \text{Ext}_{G_1}^1(L(\lambda_0), K_{i-1})$. Minimality gives us for $\mu_1 \in X_1$ that $\text{Ext}_{G_1}^i(L(\lambda_0), L(\mu_0)) \cong \text{Hom}_{G_1}(L(\lambda_0), K_i) \cong \text{Hom}_{G_1}(L(\lambda_0), I_i)$ for $i \leq r$.

We now have a G -resolution:

$$0 \rightarrow L(\mu) \rightarrow I_0 \otimes L(\mu_1)^F \xrightarrow{\partial_0} I_1 \otimes L(\mu_1)^F \xrightarrow{\partial_1} \dots$$

where $\partial_i = \delta_i \otimes \text{id}$, as tensoring is exact. Also note that such a resolution stays injective as a G_1 -resolution as $L(\mu_1)^F$ is trivial as a G_1 -module.

Now consider the E_0 -page of the LHS spectral sequence that converges to $\text{Ext}_G^*(L(\lambda), L(\mu))$ as constructed in [Par07, §2]

$$E_0^{mn} = \text{Hom}_{G/G_1}(k, \text{Hom}_{G_1}(L(\lambda), I_n \otimes L(\mu_1)^F) \otimes J_m^F)$$

where we have a G -injective resolution of the trivial module:

$$0 \rightarrow k \rightarrow J_0 \rightarrow J_1 \rightarrow \dots$$

and this spectral sequence has E_1 and E_2 page

$$\begin{aligned} E_1^{mn} &= \text{Hom}_{G/G_1}(k, \text{Ext}_{G_1}^n(L(\lambda), L(\mu)) \otimes J_m^F) \\ E_2^{mn} &= H^m(G/G_1, \text{Ext}_{G_1}^n(L(\lambda), L(\mu))). \end{aligned}$$

Consider the induced maps ∂_m^* in the following complex, which has homology $\text{Ext}_{G_1}^*(L(\lambda), L(\mu))$:

$$\text{Hom}_{G_1}(L(\lambda), I_0 \otimes L(\mu_1)^F) \xrightarrow{\partial_0^*} \text{Hom}_{G_1}(L(\lambda), I_1 \otimes L(\mu_1)^F) \xrightarrow{\partial_1^*} \dots$$

Now

$$\begin{aligned} \text{Ext}_{G_1}^m(L(\lambda), L(\mu)) &\cong \text{Ext}_{G_1}^m(L(\lambda_0), L(\mu_0)) \otimes L(\mu_1)^F \otimes L(\lambda_1^*)^F \\ &\cong \text{Hom}_{G_1}(L(\lambda_0), I_m) \otimes L(\mu_1)^F \otimes L(\lambda_1^*)^F \cong \text{Hom}_{G_1}(L(\lambda), I_m \otimes L(\mu_1)^F) \end{aligned}$$

for $m \leq r$. Thus all the differentials ∂_m^* for $m \leq r$ must be zero.

Now by [Ben98, §3.2, §3.4] we know that the spectral sequence can be constructed using derived couples. We have

$$\begin{aligned} D_0^{mn} &= \bigoplus_{m+n=e+f, e \geq m} E_0^{ef} \\ E_1^{mn} &= H(E_0^{mn}, d_0) \\ D_1^{mn} &= H(E_0^{mn} \oplus E_0^{m+1, n-1} \oplus \dots, d_0 + d_1) \end{aligned}$$

We define the higher derived couples by taking the derived couple of the previous one. We have an exact diagram of doubly graded k -modules

$$\begin{array}{ccc} D_l & \xrightarrow{i_l} & D_l \\ & \swarrow k_l \quad \searrow j_l & \\ & E_l & \end{array}$$

4

The derived couple (for $l \geq 1$) is defined by

$$\begin{aligned} D_{l+1}^{mn} &= \text{im } i_l^{m+1, n-1} \subseteq D_l^{mn} & E_{l+1}^{mn} &= H(E_l^{mn}, d_l) \\ i_{l+1}^{mn} &= i_l^{mn} \Big|_{D_{l+1}} & j_{l+1}^{mn}(i_l^{m+1, n-1}(x)) &= j_l^{m+1, n-1}(x) + \text{im}(d_l) \\ k_{l+1}^{mn}(z + \text{im}(d_l)) &= k_l^{mn}(z) & d_{l+1} &= j_{l+1} \circ k_{l+1} \end{aligned}$$

And the degrees of the maps k , j and d are:

$$\deg(i_n) = (-1, 1), \quad \deg(j_n) = (n-1, n+1), \quad \deg(k_n) = (1, 0).$$

Now using [Par07, Lemma 2.1] we have that $d_0^{mn} = 0$ implies that $k_2^{m-1, n+1} = 0$. Thus since $d_0^{mn} = 0$ for $m \leq r$ we have $k_2^{mn} = 0$ for $m \leq r-1$. Thus all $k_l^{mn} = 0$ for all $l \geq 2$ and $m \leq r-1$. As $d_l^{mn} = j_l^{m+1, n} \circ k_l^{mn}$ we also get $d_l^{mn} = 0$ for all $l \geq 2$ and $m \leq r-1$.

In other words, as all these differentials are zero on the E_2 page and remain zero, the terms E_2^{mn} with $m \leq r-1$ must already be the stable value. That is, $E_\infty^{mn} = E_2^{mn}$ for $m \leq r-1$.

This easily gives us that

$$\text{Ext}_G^i(L(\lambda), L(\mu)) = \bigoplus_{j=0}^i E_2^{i-j, j}$$

for $i < r$. To get the result for $i = r$, we note that all the terms in the sum

$$\bigoplus_{j=0}^r E_\infty^{r-j, j}$$

stabilise at the E_2 page by the above, except, possibly the term $E_\infty^{r, 0}$. But here clearly $E_\infty^{r, 0} = E_2^{r, 0}$ as all incoming differentials are zero by the above, and the leaving differential $d_l^{r, 0}$ is always zero as our spectral sequence is first quadrant.

We may similarly argue for $r+1$. We consider

$$\bigoplus_{j=0}^{r+1} E_\infty^{r+1-j, j}.$$

As before all terms except possibly $E_\infty^{r, 1}$ and $E_\infty^{r+1, 0}$ stabilise at the E_2 page. The same argument as in the previous case gives $E_\infty^{r+1, 0} = E_2^{r+1, 0}$.

Now note that all incoming differentials to $E_l^{r, 1}$ are zero for $l \geq 2$ by the above. We also have that $d_l^{r, 1} = 0$ for $l \geq 3$, again since the spectral sequence is first quadrant. So we need only check that $d_2^{r, 1} = 0$, but this is true using [Don82, Main Theorem]. Thus we also get the result for $r+1$. \square

Acknowledgements. The authors wish to thank Len Scott and Dan Nakano for comments on a previous version of this paper.

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